JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 5, September-October 1992

Application of a Game Theoretic Controller to a Benchmark Problem

Ihnseok Rhee* and Jason L. Speyer†
University of Texas at Austin, Austin, Texas 78712

The game theoretic controller whose structure is identical to that of both the linear exponential Gaussian and the H_∞ controller is applied to the problem of controlling a mass-spring system that approximates the dynamics of a flexible structure. By viewing the plant parameter variation as an internal feedback loop, plant uncertainties of the system, input, and output matrices can be decomposed into a fictitious input/output system with unknown gains. These fictitious input/output directions due to parameter uncertainty are used in constructing the gains for the game theoretic controller. The resulting control reduces the effect of parameter uncertainty on the system performance.

I. Introduction

A SYNTHESIS procedure is described for the design of a state feedback control law for a linear time-invariant system in the presence of parameter uncertainty in the system, input, and output matrices. The parameter uncertainty is modeled via an input/output decomposition procedure.[1,2] A differential game approach has been taken for this problem in Ref. 3, where the parameter uncertainty was not decomposed and only the uncertainty in the system matrix is considered. In Refs. 4-6 the Lyapunov stability theory has been used to design a control law for a system with uncertainty. In Refs. 1 and 2, by adopting an input/output decomposition of the parameter uncertainty, the uncertain system is represented as an internal feedback loop (IFL) in which the parameter uncertainty is embedded in the system as a fictitious disturbance. Tahk and Speyer[1,2] developed the parameter robust linear quadratic Gaussian (PRLQG) synthesis procedure, which is an LQG design based on an extension of loop-transfer recovery for the IFL description. In Refs. 1, 2, and 6, the system is augmented to accommodate the input and output matrix uncertainty. In this paper, by considering the input and a fictitious input in the IFL description as two noncooperative players, a finite-time linear differential game problem is constructed based on the results of Ref. 7. By taking the limit to an infinite-time, time-invariant linear system, a time-invariant control law is obtained. It is shown that the resulting time-invariant controller stabilizes the uncertain system for a prescribed parameter uncertainty bound. These results are presented in Sec. II.

This approach is applied in Sec. III to a benchmark problem composed of two masses and a spring with an unknown spring constant. The input is applied to the first mass and a noisy measurement is made of the position of the second mass. Furthermore, a harmonic forcing function of unknown amplitude and phase is applied to the second mass. The objective is to regulate the second mass about the zero position given the assumed uncertainties. A robust compensation is determined that has four nonminimal phase zeros.

II. Game Theoretic Controller

A controller for a linear time-invariant system with parameter uncertainties in the system, input, and output matrices is derived via the differential game framework.

Consider a time-invariant system with uncertainties in system, input, and output matrices described by

$$\dot{x} = (A_0 + \Delta A)x + (B_0 + \Delta B)u \tag{1}$$

$$z = (H_0 + \Delta H)x \tag{2}$$

where x, u, and z denote the state vector, the input vector, and the measurement vector, respectively; A_0, B_0, and H_0 denote the nominal system matrix, the nominal input matrix, and the nominal measurement matrix with suitable dimensions, respectively; and ΔA, ΔB, and ΔH are perturbations of the system matrix, the input matrix, and the measurement matrix, respectively, due to parameter variations. It is assumed that (A_0, B_0) is a stabilizable pair and (H_0, A_0) is a detectable pair.

By adopting the input/output decomposition modeling[1,2] of the perturbations, ΔA, ΔB, and ΔH are represented as

$$\Delta A = DL_a(\epsilon)E, \qquad \Delta B = FL_b(\epsilon)G, \qquad \Delta H = YL_h(\epsilon)Z \tag{3}$$

where ϵ denotes the parameter variation vector, which is constant but unknown, and all other matrices are known constant matrices. The elements of ϵ need not be independent of each other.

A. State Feedback

In this subsection all states are assumed to be perfectly measured, and the control u is restricted to a state feedback, i.e., $u = u(x)$.

With the plant perturbation modeling given by Eq. (3), the uncertain dynamic system [Eq. (1)] can be represented as an internal feedback loop[1,2] in which the system is assumed to be forced by fictitious disturbances caused by the parameter uncertainty:

$$\dot{x} = A_0x + B_0u + \Gamma_f w_f \tag{4}$$

$$y_f = \begin{bmatrix} E \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ G \end{bmatrix} u \tag{5}$$

$$w_f = L(\epsilon)y_f \tag{6}$$

where $\Gamma_f = [D\ F]$, w_f is the fictitious disturbance, and

$$L(\epsilon) = \begin{bmatrix} L_a(\epsilon) & 0 \\ 0 & L_b(\epsilon) \end{bmatrix}$$

Received March 22, 1991; revision received Oct. 4, 1991; accepted for publication Oct. 4, 1991.

*Currently, Senior Research Engineer, Korea Aerospace Research Institute, Daeduck Science Town, Daejum, Korea.

†Harry H. Power Professor of Engineering; currently, Professor, Mechanical, Aerospace and Nuclear Engineering Department, University of California, Los Angeles, CA 90024.

Consider a quadratic performance index,

$$J = \frac{1}{2}\int_0^T \|y\|^2 \, dt$$

where T is a fixed final time, and

$$y = \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ C_1 \end{bmatrix} u$$

Assume that all admissible parameter variations are characterized as

$$\frac{\int_0^T y_f^T L^T(\epsilon) L(\epsilon) y_f^T \, dt}{\int_0^T y_f^T y_f \, dt} = \frac{\int_0^T w_f^T w_f \, dt}{\int_0^T y_f^T y_f \, dt} \le \gamma^2$$

where γ is a positive constant. Note that $\|L(\epsilon)\|_s \le \gamma$ for all admissible parameter variations, where $\| \cdot \|_s$ denotes the spectral norm.

For a given control law $u = u(x)$, the performance index J achieves its maximum when the parameter variations are the worst case for the control u. The worst case occurs when w_f uses all of the available control, i.e.,

$$\gamma^{-2}\int_0^T w_f^T w_f \, dt = \int_0^T y_f^T y_f \, dt \tag{7}$$

Consider a control law that minimizes J for the worst case w_f. Then a game situation arises such that

$$\min_u \max_{w_f} J$$

subject to Eqs. (1) and (7). Adjoining the constraint (7) to the performance index J yields the problem of

$$\min_u \max_{w_f} \frac{1}{2}\int_0^T [\rho^2 y^T y + (y_f^T y_f - \gamma^{-2} w_f w_f)] \, dt \tag{8}$$

subject to Eq. (4), where ρ is a constant to be determined by trial and error to satisfy Eq. (7). It is well known[3,8] that if there exists a real symmetric solution $\Pi(t)$ over the interval $t \in [0, t_f]$ to the Riccati differential equation (RDE),

$$-\dot{\Pi} = A_0^T \Pi + \Pi A_0 - \Pi(B_0 R^{-1} B_0^T - \gamma^2 \Gamma_f \Gamma_f^T)\Pi + Q_s$$

with the final condition $\Pi(t_f) = 0$, where

$$Q_s = \rho^2 C^T C + E^T E, \qquad R = \rho^2 C_1^T C_1 + G^T G$$

and R is assumed to be positive definite, then the optimal strategies u^* and w_f^* for u and w_f, respectively, are given as

$$u^* = -R^{-1} B_0^T \Pi(t) x$$

$$w_f^* = \gamma^2 \Gamma_f^T \Pi(t) x$$

For the case where $T \to \infty$, if there exists a nonnegative definite solution to the algebraic Riccati equation (ARE),

$$0 = A_0^T \Pi_s + \Pi_s A_0 - \Pi_s (B_0 R^{-1} B_0^T - \gamma^2 \Gamma_f \Gamma_f^T)\Pi_s + Q_s \tag{9}$$

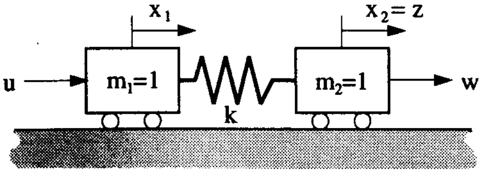

Fig. 1 Mass-spring system.

then $\Pi(t)$ converges to the minimal nonnegative definite solution[5] to the ARE (9).[3,7] Hence, u^* and w_f^* become time-invariant strategies described by

$$u^* = -R^{-1} B_0^T \bar{\Pi}_s x \tag{10a}$$

$$w_f^* = \gamma^2 \Gamma_f^T \bar{\Pi}_s x \tag{10b}$$

where $\bar{\Pi}_s$ is the minimal nonnegative definite solution to the ARE [Eq. (9)].

In the worst case design, since the fictitious disturbance w_f is not an intelligent player, only the control strategy for the control u given by Eq. (10a) can be implemented.

Claim 1. *Suppose that* $\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{G} > 0$ *and let* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *be arbitrary positive-definite matrices with suitable dimension. Then*

$$\mathcal{D}^T\mathcal{U}_1\mathcal{D} + \mathcal{G}^T\mathcal{U}_2\mathcal{G} > 0$$

Proof. It is sufficient to prove that $\mathcal{D}^T\mathcal{U}_1\mathcal{D} + \mathcal{G}^T\mathcal{U}_2\mathcal{G}$ is nonsingular. Suppose that there exists a nonzero z such that $z^T(\mathcal{D}^T\mathcal{U}_1\mathcal{D} + \mathcal{G}^T\mathcal{U}_2\mathcal{G})z = 0$. Then $\mathcal{D}z = 0$ and $\mathcal{G}z = 0$ since $\mathcal{U}_1$ and $\mathcal{U}_2$ are positive definite; hence, $(\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{G})z = 0$, which contradicts the assumption. ■

Claim 2. *Let* $\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{G} = \mathcal{F}^T\mathcal{F}$ *and let* $\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{U}\mathcal{G} = \mathcal{F}_1^T\mathcal{F}_1$, *where* $\mathcal{U}$ *is an arbitrary positive-definite matrix with a suitable dimension. If* $(\mathcal{F}, \mathcal{A})$ *is detectable, then* $(\mathcal{F}_1, \mathcal{A} + \mathcal{X}\mathcal{G})$ *is detectable for all* $\mathcal{X}$ *with suitable dimensions.*

Proof. Suppose that $(\mathcal{F}_1, \mathcal{A} + \mathcal{X}\mathcal{G})$ is not detectable. Then there exists a nonzero vector z for some s in the closed right half plane such that $(sI - \mathcal{A} - \mathcal{X}\mathcal{G})z = 0$ and $\mathcal{F}_1 z = 0$. Since $\mathcal{U} > 0$,

$$z^T(\mathcal{F}_1\mathcal{F}_1)z = z^T(\mathcal{D}^T\mathcal{D} + \mathcal{G}^T\mathcal{U}\mathcal{G})z = 0$$

which implies that $\mathcal{D}z = 0$ and $Gz = 0$. Hence,

$$(sI - \mathcal{A} - \mathcal{X}\mathcal{G})z = (sI - \mathcal{A})z$$

Therefore,

$$\begin{bmatrix} sI - \mathcal{A} \\ \mathcal{F} \end{bmatrix} z = 0$$

which contradicts the assumption that $(\mathcal{F}, \mathcal{A})$ is detectable. ■

Proposition 1. *Assume that* $R > 0$ *and* $(Q_s^{1/2}, A_0)$ *is a detectable pair. Suppose that for a given* ρ *and* γ *there exists a nonnegative definite solution* Π_s *to the ARE* [Eq. (9)]. *Then the control law given as*

$$u = -R^{-1} B_0^T \Pi_s x \tag{11}$$

stabilizes the uncertain dynamic system (1) for all ϵ *such that* $\|L_a(\epsilon)\|_s < \gamma$ *and* $\|L_b(\epsilon)\|_s < \gamma$.

Proof. By using the control law (11), the closed-loop system is described as

$$\dot{x} = A_s x \tag{12}$$

where

$$A_s = A_0 + DL_a(\epsilon)E - \{B_0 + FL_b(\epsilon)G\}R^{-1}B_0^T \Pi_s$$

The ARE [Eq. (9)] can be rewritten as following the Lyapunov equation:

$$A_s^T \Pi_s + \Pi_s A_s = -Q_1 \tag{13}$$

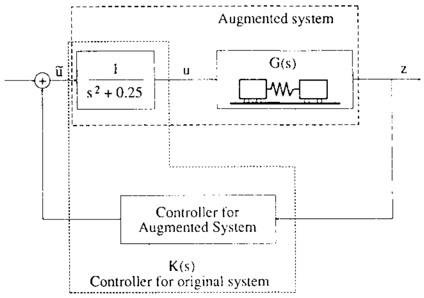

Fig. 2 Block diagram of closed-loop system.

where

$$Q_1 = \Pi_s B_0 R^{-1}\Delta_b R^{-1} B_0^T \Pi_s + E^T\Delta_a E + \rho^2 C^T C$$
$$+ \gamma^2(\Pi_s D - \gamma^{-2}E^T L_a^T)(\Pi_s D - \gamma^{-2}E^T L_a^T)^T$$
$$+ \gamma^{-2}\Pi_s(\gamma^2 F + B_0 R^{-1} G^T L_b^T)(\gamma^2 F + B_0 R^{-1} G^T L_b^T)^T \Pi_s$$
$$\Delta_a = I - \gamma^{-2} L_a(\epsilon)^T L_a(\epsilon)$$
$$\Delta_b = \rho^2 C_1^T C_1 + G^T[I - \gamma^{-2} L_b(\epsilon)^T L_b(\epsilon)]G$$

$\|L_a(\epsilon)\|_s < \gamma$ implies that $\Delta_a > 0$, and $\|L_b(\epsilon)\|_s < \gamma$ and Claim 1 yield $\Delta_b > 0$. Hence, Q_1 is nonnegative definite. Since $(Q_s^{1/2}, A_0)$ is detectable by assumption, it follows from Claim 2 that $[(E^T\Delta_a E + \rho^2 C^T C)^{1/2},\ A_0 + DL_a E]$ is detectable. From Lemma 4.1 (Ref. 9), $[(\Pi_s B_0 R^{-1}\Delta_b R^{-1} B_0^T \Pi_s + E^T \Delta_a E + \rho^2 C^T C)^{1/2},\ A_s]$ is also a detectable pair. Applying Lemma 4.1 of Ref. 9 again yields that $(Q_1^{1/2}, A_s)$ is detectable. Applying Lemma 4.2 of Ref. 9 to the Lyapunov equation [Eq. (13)] completes the proof. ■

Note that Proposition 1 holds for any nonnegative solution to the ARE [Eq. (9)]. However, the minimal nonnegative solution $\bar{\Pi}_s$ produces the smallest gain for the control law.

To design the controller (11), the design parameters ρ and γ should be chosen for the ARE [Eq. (9)] to have a nonnegative definite solution. In particular, as the value of ρ increases, system performance improves, whereas as the value of γ increases, stability robustness with respect to parameter variation improves.

B. Measurement Feedback

In this subsection the state is assumed to be partially measured by Eq. (2) and the control u is restricted to be a measurement feedback.

By use of the uncertainty modeling of Eq. (3), system (1) and (2) include an IFL description:

$$\dot{x} = A_0 x + B_0 u + \Gamma_f w_f \tag{14}$$

$$z = H_0 x + Y v_f \tag{15}$$

$$y_f = \begin{bmatrix} E \\ 0 \\ Z \end{bmatrix} x + \begin{bmatrix} 0 \\ G \\ 0 \end{bmatrix} u \tag{16}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ v_f \end{bmatrix} = \begin{bmatrix} L_a(\epsilon) & 0 & 0 \\ 0 & L_b(\epsilon) & 0 \\ 0 & 0 & L_h(\epsilon) \end{bmatrix} y_f \tag{17}$$

where $\Gamma_f = [D\ F]$ and $w_f = [w_1^T\ w_2^T]^T$.

With an approach similar to that taken in Sec. II.A, a differential game, where the fictitious disturbances w_f and v_f and the initial conditions play against the control u, is constructed such that

$$\min_u \max_{w_f} \max_{v_f} \max_{x(0)} [-\rho_1^2\{x(0) - \hat{x}_0\}^T\{x(0) - \hat{x}_0\}$$
$$+ \int_0^T \{\rho^2 y^T y + y_f^T y_f - \gamma^{-2}(w_f^T w_f + v_f^T v_f)\}\, dt]$$

subject to Eqs. (14) and (15), where the cost for the initial conditions is included to handle the uncertainty in the initial condition from the nominal value of $\hat{x}_0$. As $T \to \infty$, a time-invariant controller is obtained in Ref. 7 as

$$\dot{x}_c = A_c x_c + B_c z \tag{18a}$$

$$u = C_c x_c \tag{18b}$$

where

$$A_c = A_0 - B_0 R^{-1} B_0^T \Pi_m - M H_0^T V^{-1} H_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m$$
$$B_c = M H_0^T V^{-1}, \qquad C_c = -R^{-1} B_0^T \Pi_m$$
$$M = (I - \gamma^2 P_m \Pi_m)^{-1} P_m$$

if there exist $\Pi_m \geq 0$ and $P_m > 0$ satisfying the AREs:

$$0 = A_0^T \Pi_m + \Pi_m A_0 - \Pi_m(B_0 R^{-1} B_0^T - \gamma^2 \Gamma_f \Gamma_f^T)\Pi_m + Q_m \tag{19}$$

$$0 = A_0 P_m + P_m A_0^T - P_m(H_0^T V^{-1} H_0 - \gamma^2 Q_m) P_m + \Gamma_f \Gamma_f^T \tag{20}$$

such that $P_m - \gamma^2 \Pi_m > 0$, where

$$Q_m = \rho^2 C^T C + E^T E + Z^T Z, \qquad V = YY^T$$

and V is assumed to be positive definite. Since both AREs [(19) and (20)] may have more than one nonnegative definite or positive definite solution, many controllers can be constructed from the formulation (18). Note that $M > 0$ and satisfies the ARE:

$$(A_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m) M + M(A_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m)^T$$
$$- M[H_0^T V^{-1} H_0 - \gamma^2 \Pi_m B_0 R^{-1} B_0^T \Pi_m] M + \Gamma_f \Gamma_f^T = 0 \tag{21}$$

By using the controller (18), the closed-loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = (A_{cl} + \Delta A_{cl}) \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{22}$$

$$A_{cl} = \begin{bmatrix} A_0 & B_0 C_c \\ B_c H_0 & A_c \end{bmatrix}, \qquad \Delta A_{cl} = \begin{bmatrix} DL_a E & FL_b G C_c \\ B_c Y L_h Z & 0 \end{bmatrix}$$

and where ΔA_{cl} is the variation of the closed-loop system due to the uncertainty in system (1) and (2).

Proposition 2. *Assume that $R > 0$, $V > 0$, $(Q_m^{1/2}, A_0)$ is detectable, and (A_0, Γ_f) is controllable. If there exist $\Pi_m \geq 0$, $P_m > 0$ such that $P_m^{-1} - \gamma^{-2}\Pi_m$, then the controller (18) stabilizes the closed-loop system (22) for all ϵ such that*

$$\|L_a(\epsilon)\|_s < \gamma, \qquad \|L_b(\epsilon)\|_s < \gamma, \qquad \|L_h(\epsilon)\|_s < \gamma \tag{23}$$

Proof. Equation (21) can be rewritten as

$$MA_1^T + A_1 M = -Q_1$$

where

$$A_1 = A_0 - MH_0^T V^{-1} H_0 + \gamma^2 \Gamma_f \Gamma_f^T \Pi_m$$

$$Q_1 = M(H_0^T V^{-1} H_0 + \gamma^2 \Pi_m B_0 R^{-1} B_0^T \Pi_m) M + \Gamma_f \Gamma_f^T$$

Since (A_0, Γ_f) is controllable by assumption, it follows from Lemma 4.1 (Ref. 9) that $(A_1, Q_1^{1/2})$ is controllable. Since $M > 0$, it follows from Lemma 4.2 (Ref. 9) that A_1 is stable. It can be verified that $\mathcal{K}$, defined as

$$\mathcal{K} = \begin{bmatrix} P_m^{-1} & -M^{-1} \\ -M^{-1} & M^{-1} \end{bmatrix}$$

satisfies the ARE:

$$0 = A_{cl}^T \mathcal{K} + \mathcal{K} A_{cl} + \mathcal{K} Q_2 \mathcal{K} + \gamma^2 Q_3 \tag{24}$$

where

$$Q_2 = \begin{bmatrix} \Gamma_f \Gamma_f^T & 0 \\ 0 & B_c V B_c^T \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} Q_m & 0 \\ 0 & C_c^T R C_c \end{bmatrix}$$

Rewriting Eq. (24) as

$$(A_{cl} + \Delta A_{cl})^T \mathcal{K} + \mathcal{K}(A_{cl} + \Delta A_{cl}) = -Q_4 \tag{25}$$

where $Q_4 = \mathcal{K} Q_2 \mathcal{K} + \gamma^2 Q_3 - \Delta A_{cl}^T \mathcal{K} - \mathcal{K} \Delta A_{cl}$. Then Q_4 can be written in the following form:

$$Q_4 = \begin{bmatrix} P_m^{-1} F \\ \Pi_m B_0 R^{-1} G^T L_b^T - M^{-1} F \end{bmatrix} \times \begin{bmatrix} P_m^{-1} F \\ \Pi_m B_0 R^{-1} G^T L_b^T - M^{-1} F \end{bmatrix}^T$$

$$+ \begin{bmatrix} E^T L_a^T - P_m^{-1} D \\ M^{-1} D \end{bmatrix} \begin{bmatrix} E^T L_a^T - P_m^{-1} D \\ M^{-1} D \end{bmatrix}^T$$

$$+ \begin{bmatrix} Z^T L_h^T + H_0^T V^{-1} Y \\ -H_0 V^{-1} Y \end{bmatrix} \begin{bmatrix} Z^T L_h^T + H_0^T V^{-1} Y \\ -H_0 V^{-1} Y \end{bmatrix}^T$$

$$+ \gamma^2 \begin{bmatrix} \rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z & 0 \\ 0 & \Pi_m B_0 R^{-1} \Delta_b R^{-1} B_0^T \Pi_m \end{bmatrix}$$

where

$$\Delta_h = I - \gamma^{-2} L_h^T L_h$$

Since $\|L_a(\epsilon)\|_s < \gamma$, $\|L_b(\epsilon)\|_s < \gamma$, and $\|L_h(\epsilon)\|_s < \gamma$, Q_4 is nonnegative definite. From Lemma 4.2 (Ref. 9),

$$(\{\rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z\}^{1/2}, A_0 + DL_a E)$$

is detectable since $(Q_m^{1/2}, A_0)$ is detectable and $\Delta_a, \Delta_h > 0$. Therefore, there exists L such that A_2, defined as

$$A_2 \triangleq A_0 + DL_a E - L(\rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z)^{1/2}$$

is stable. Let

$$Q_5 = \begin{bmatrix} (\rho^2 C^T C + E^T \Delta_a E + Z^T \Delta_h Z)^{1/2} & 0 \\ 0 & \Delta_b^{1/2} R^{-1} B_0^T \Pi_m \end{bmatrix}$$

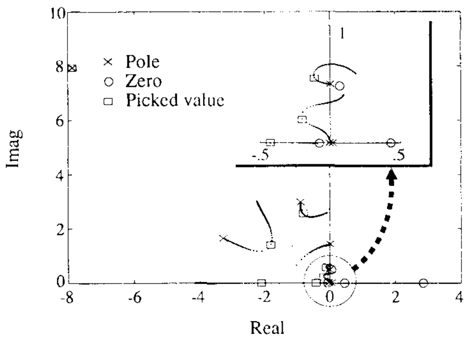

Fig. 3 Root locus of $1 - \alpha G(s)K(s)$ with respect to α.

Since $I - \gamma^{-2} L_b^T L_b > 0$ and $R > 0$, it follows from Claim 1 that $\Delta_b > 0$. Observe that, for this L,

$$A_{cl} + \Delta A_{cl} - \begin{bmatrix} L & -(B_0 + FL_b G)\Delta_b^{-1/2} \\ 0 & -B_0 \Delta_b^{-1/2} \end{bmatrix} Q_5$$

$$= \begin{bmatrix} A_2 & 0 \\ MH_0^T V^{-1}(H_0 + YL_h Z) & A_1 \end{bmatrix}$$

Since A_1 and A_2 are stable, $(Q_5, A_{cl} + \Delta A_{cl})$ is a detectable pair which implies that, by Lemma 4.1 of Ref. 9, $(Q_4, A_{cl} + \Delta A_{cl})$ is detectable. Since $\mathcal{K}$ is a nonnegative definite matrix, the proof is completed by applying Lemma 4.2 of Ref. 9 to Eq. (25). ■

Note that the proposition holds for all controllers constructed from the solution of AREs and is therefore very conservative.

To design the controller (18), the design parameters ρ and γ should be choosen for the AREs [(19) and (20)] to have a nonnegative definite solution and a positive definite solution, respectively. In particular, as the value of ρ increases, system performance improves, whereas as the value of γ increases, stability robustness with respect to parameter variation improves.

In the usual case the positivity of V and the controllability of (A, Γ_f) do not hold. However, these can be avoided by redefining V and Γ_f as

$$V = \Gamma_1 \Gamma_1^T + YY^T, \qquad \Gamma_f = [\Gamma \;\; D \;\; F]$$

where Γ_1 and Γ are chosen to ensure that $V > 0$ and (A, Γ_f) is controllable. It can be proved with minor change that Proposition 2 holds for these new V and Γ_f.

III. Two Mass-Spring System

Consider a mass-spring system, shown in Fig. 1, that approximates the dynamics of a flexible structure.[2] The system is described by

$$\ddot{x}_1 + k(x_1 - x_2) = u \tag{26a}$$

$$\ddot{x}_2 + k(x_2 - x_1) = w \tag{26b}$$

with a noncollocated measurement

$$z = x_2 \tag{27}$$

where k is an unknown constant with nominal value $k_0 = 1$, u is an actuator input, and w is a cyclic disturbance described by

$$w(t) = A_w \sin(0.5t + \varphi)$$

where A_w and φ are constant but unknown. The transfer function form of the system and measurement equations, Eqs. (26) and (27), respectively, is given as

$$G(s) \triangleq \frac{z(s)}{u(s)} = \frac{1}{s^2(s^2+2)}$$

The design objective is to regulate x_2 and to reject the external cyclic disturbance in x_2 for all k with $0.5 < k < 2$.

To handle the cyclic disturbance, differentiate Eq. (26) until w disappears in the resulting system. Differentiating Eq. (26) twice yields

$$x_1^{(IV)} = -k(\ddot{x}_1 + 0.25x_1 - \ddot{x}_2 - 0.25x_2) - 0.25\ddot{x}_1 + \tilde{u} \tag{28a}$$

$$x_2^{(IV)} = -k(\ddot{x}_2 + 0.25x_2 - \ddot{x}_1 - 0.25x_1) - 0.25\ddot{x}_2 \tag{28b}$$

where the parenthetical superscripts represent the time-derivative order and $\tilde{u}$ is a new control variable defined as

$$\tilde{u} = \ddot{u} + 0.25u \tag{29}$$

The variation of system matrix due to the uncertainty of k can be decomposed as

$$\Delta A = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{D} \begin{bmatrix} \dfrac{\Delta k}{n} \end{bmatrix} \underbrace{\begin{bmatrix} -n & 0 & 0.25n & 0 & n & 0 \end{bmatrix}}_{E}$$

With choices of

$$C = H, \qquad \Gamma = 0.07 \cdot [0\ 1\ 0\ 0\ 0\ 1]^T, \qquad \Gamma_1 = 0.033$$

$$C_1 = 0.08, \qquad \rho = 1, \qquad \gamma = 0.043, \qquad n = 10$$

the control $\tilde{u}$ can be obtained by using Eq. (18) as

$$\dot{x}_c = A_c x_c + B_c z, \qquad \tilde{u} = C_c x_c \tag{31}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 8.41 & 0 & 0 & 0 \\ -147.26 & -17.11 & 60.32 & -61.00 & 44.21 & -174.52 \\ 0 & 0 & -7.19 & 1 & 0 & 0 \\ 0 & 0 & -25.78 & 0 & 1 & 0 \\ 0 & 0 & -51.65 & 0 & 0 & 1 \\ 1.03 & 0 & -41.20 & 0.11 & -1.11 & 0.30 \end{bmatrix}$$

$$B_c = [-8.41\ \ -37.85\ \ 7.19\ \ -25.78\ \ 51.65\ \ 40.40]^T$$

$$C_c = [-146.23\ \ -17.11\ \ 22.24\ \ -60.89\ \ 43.34\ \ -174.22]$$

The new system (28) contains uncontrollable poles at $s = \pm 0.5j$. To remove the uncontrollable poles from Eqs. (28), a new state, ξ, is introduced as $\xi = \ddot{x}_1 + 0.25x_1$. Then Eqs. (28) are represented in terms of x_2 and ξ as

$$\ddot{\xi} = -k\xi + k(\ddot{x}_2 + 0.25x_2) + \tilde{u} \tag{30a}$$

$$x_2^{(IV)} = -(k+0.25)\ddot{x}_2 - 0.25kx_2 + k\xi \tag{30b}$$

A controller is designed for this augmented system. Figure 2 shows that the controller for original system is constructed by combining the controller for the augmented system (30) and the relation (29). Define

$$x = [\xi\ \dot{\xi}\ x_2\ \dot{x}_2\ \ddot{x}_2\ x_2^{(III)}]^T$$

Then Eqs. (27) and (30) can be represented in state-space form as

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k & 0 & 0.25k & 0 & k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k & 0 & -0.25k & 0 & -k-0.25 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{B} \tilde{u}$$

$$z = \underbrace{[0\ 0\ 1\ 0\ 0\ 0]}_{H} x$$

Note that n is a weighting between the E direction and C, and Γ is chosen to ensure that (A, Γ_f) is controllable (see Proposition 2). The design parameters ρ, γ, and n were chosen to satisfy the robustness requirement that $0.5 < k < 2$ and the transient requirement that the system settle within 20 s. The minimal nonnegative definite solutions for the AREs [(19) and (20)] are used in controller design. Combining Eqs. (29) and (31) yields an eighth-order controller for the original system (28) in the form of

$$\dot{\tilde{x}}_c = \begin{bmatrix} A_c & 0_{6\times 2} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} C_c & \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix} \end{bmatrix} \tilde{x}_c + \begin{bmatrix} B_c \\ 0_{2\times 1} \end{bmatrix} z$$

$$u = [0_{1\times 6}\ \ 1\ \ 0]\tilde{x}_c$$

or in the transfer function form:

$$K(s) \triangleq \frac{u(s)}{z(s)} = \frac{-4430(s+0.08)(s-0.44)(s-2.83)}{(s^2+0.25)(s^2+1.78s+9.67)} \times \frac{(s^2-0.1s+0.24)}{(s^2+6.56s+13.51)(s^2+15.68s+124.99)}$$

where the compensator poles are at $\pm 0.5j$, $-7.84 \pm 7.97j$, $-3.28 \pm 1.66j$, and $-0.89 \pm 2.98j$, and the complex compensator zeros are at $0.05 \pm 0.49j$. The zero configuration of the compensator represents nonminimum compensation. The closed-loop poles are at -2.14, -0.33, $-0.13 \pm 0.56j$, $-0.19 \pm 0.25j$, $-0.77 \pm 2.50j$, $-1.83 \pm 1.45j$, and $-7.84 \pm 7.97j$.

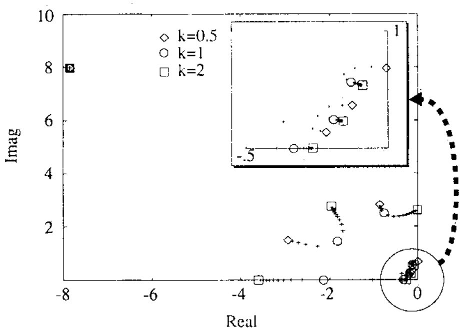

Fig. 4 Root locus of closed-loop system with respect k.

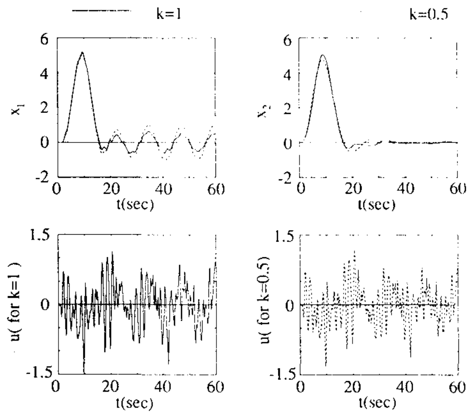

Fig. 5 Time response.

Figure 3 shows the root locus of $1 - \alpha G(s)K(s)$ with respect to α. For the controller $K(s)$, the gain margin is 3.23 dB at the frequency 0.67 rad/s, and the phase margin is 25 deg at the frequency 0.20 rad/s. The values of γ and n show that the stability margin $|\Delta k| < 0.43$ is guaranteed for this compensator. However, the root locus, shown in Fig. 4, shows that the compensator stabilizes the system over $0.5 < k < 2$. Note that the compensator pole at $-7.84 \pm 7.97j$ is unaffected by parameter or open-loop gain changes. The removal of these poles from the compensator does not affect stability robustness or transient response.

In the simulation the measurement is assumed corrupted by zero mean white Gaussian noise with a power spectral density of (0.33).[2] Time responses, shown in Fig. 5, for the nominal system and the perturbed system with $k = 0.5$ are simulated. For both simulations, $A_w = 0.5$, $\varphi = 0$, and all initial conditions are zero. Figure 5 shows that, for the nominal case, the controlled variable x_2 has settled down and the cyclic disturbance is rejected in x_2 in ~ 20 s, and for the perturbed system with $k = 0.5$, the settling time has been delayed.

IV. Conclusions

A game theoretic controller was applied to a mass-spring system disturbed by cyclic external force. The cyclic disturbance was augmented to the system by a procedure involving differentiation and transformation. The resulting state and control are used to design the game theoretic compensator. A nonminimum phase compensator resulted.

Acknowledgment

This work was supported in part by the Air Force Office of Scientific Research under Grant AFOSR 91-0077.

References

[1]Tahk, M., and Speyer, J. L., "Modelling of Parameter Variations and Asymptotic LQG Synthesis," *IEEE Transactions on Automatic Control,* Vol. AC-32, No. 9, Sept. 1987, pp. 793–801.

[2]Tahk, M., and Speyer, J. L., "Parameter Robust Linear-Quadratic-Gaussian Design Synthesis with Flexible Structure Control Applications," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 4, 1989, pp. 460–468.

[3]Medanic J., "Bounds on the Performance Index and the Riccati Equation in Differential Games," *IEEE Transactions on Automatic Control,* Vol. 12, No. 5, Oct. 1967, pp. 613–614.

[4]Petersen, I. R., and Hollot, C. V., "A Riccati Equation Approach to the Stabilization of Uncertain Linear Systems," *Automatica,* Vol. 22, 1986, pp. 397–411.

[5]Petersen, I. R., "Some New Results on Algebraic Riccati Equations Arising in Linear Quadratic Differential Games and the Stabilization of Uncertain Linear Systems," *Systems and Control Letters,* Vol. 10, 1988, pp. 341–348.

[6]Zhou, K., and Khargonekar, P. P., "Robust Stabilization of Linear System with Norm-bounded Time-Varying Uncertainty," *Systems and Control Letters,* Vol. 10, 1988, pp. 17–20.

[7]Rhee, I., and Speyer, J. L., "A Game Theoretic Approach to a Finite-Time Disturbance Attenuation Problem," *IEEE Transactions on Automatic Control,* Vol. 36, No. 9, Sept. 1991.

[8]Mageirou, E. F., "Values and Strategies for Infinite Time Linear Quadratic Games," *IEEE Transactions on Automatic Control,* Vol. 21, No. 4, Aug. 1976, pp. 547–550.

[9]Wonham, W. M., "On a Matrix Riccati Equation of Stochastic Control," *SIAM Journal of Control,* Vol. 6, No. 4, 1968, pp. 681–697.